Professor of Mathematics in the Imperial University, Tokio, and now its president. Member of the Japanese House of Lords. Author of a Geometry in English and Japanese].

1878. Sargant, 7th Wrangler.

1878. Leverson, 15th Wrangler.

1879. Karl Pearson, 3rd Wrangler. [The celebrated writer on the mathematics of Evolution and Darwinism].

1883. Romer, H. S., 20th Wrangler. [Brother to Ld. Romer, Senior Wrangler in 1863].

1885. Berry, A., Senior Wrangler and 2d Smith's Prizeman.

1886. Hooker, J. H., 29th Wrangler.

1887. Norris, J. R., 16th Wrangler.

1890. Bennett, G. T., Senior Wrangler and First Smith's Prizeman.

1890. Vaughan, A., Bracketed 3rd Wrangler.

1892. Kirby, S. F., 12th Wrangler.

1892. Clay, R. G., 21st Wrangler.

1895. Schroder, H. M., 29th Wrangler.

F. W. Frankland, the eminent writer on non-Euclidean Geometry (now an Actuary in New York) was a favorite pupil.

Lady Gwendolen Cecil, daughter of Lord Salisbury, read three seasons with Mr. Tucker, showing fine mathematical abilities.

A mere list of the writings of Mr. Tucker would give no adequate idea of their value. We may mention almost at random his Appendix to the Proceedings of the London Mathematical Society No. 279 containing Conjugate "Tucker" Circles. The Index to the Proceedings gives the titles of fifteen papers, ending with "Some Properties of Two Tucker-Circles."

In 1883 Mr. Tucker rediscovered Lemoine's circle and wrote a paper on it in the Quarterly Journal under the title: The "Triplicate-Ratio" Circle; which may be said to have started the general English interest in this new development of geometry.

Mr. Tucker did a fine piece of work in editing Clifford's Papers and his Dynamics Part II.

All the world will rejoice that as a result of a petition sent in to Mr. Balfour by the principal mathematicians of England, the Queen has granted Mr. Tucker a Civil List pension of forty pounds a year. The money is little, but such recognition of services must be highly gratifying to a loyal Briton.

DIAGRAMMATIC PROOF OF THE CONDITION OF FUNCTIONALITY IN COMPLEX FUNCTIONS.

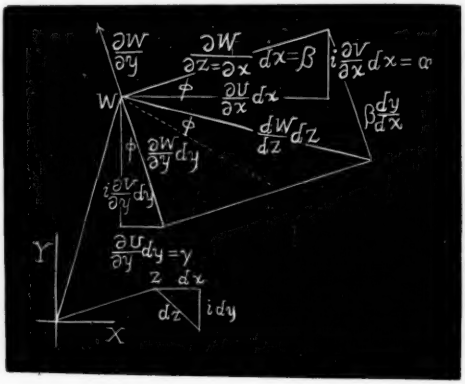
By A. LATHAM BAKER, C. E., Ph. D., University of Rochester, Rochester, N. Y.

1°. If the given complex function $W(z)=W(x+iy)=U+iV$ has a derivative we may write $\frac{dW}{dz}=w$, which phrased for the Argand Diagram is: *The triangle determined by dW, dz is similar to that determined by w, 1.*

At the point z, suppose z to take the increment $\partial z=dx$, causing in W, the change

$$\frac{\partial W}{\partial z=dx}dx=\frac{\partial U}{\partial x}dx+i\frac{dV}{\partial x}dx$$

lying at an angle, say ϕ, with $\partial z=dx$. This is shown in the diagram.



Suppose z had taken the increment dz, causing in W the change $\frac{dW}{dz}dz$, making with dz the same angle ϕ.

Since $\frac{\partial W}{\partial x}=\frac{dW}{dz}=\frac{w}{1}$, we have $\frac{dW}{\partial W}=\frac{dz}{\partial x}$ and $\frac{dW}{dz}dz$ will make with $\frac{\partial W}{\partial x}dx$ an angle whose secant is $\frac{dz}{dx}=\tan^{-1}\frac{dy}{dx}$.

Similarly when z takes the increment idy, $\frac{\partial W}{\partial z=dy}dy$ makes with the vertical the same angle ϕ. All these increments are shown in the diagram.

2°. Denoting the *lengths* of the lines in the diagram by Greek letters as shown, we get

$$\gamma=\beta\frac{dy}{dx}\,.\sin\beta=\beta\frac{dy}{dx}\,.\frac{\alpha}{\beta}=\alpha\frac{dy}{dx};$$

therefore

$$\frac{\gamma}{\alpha}=\left\{\frac{\frac{\partial U}{\partial y}dy}{\frac{\partial V}{\partial x}dx}\right\}=\frac{dy}{dx},$$

or introducing the Argand elements,

$$\frac{\frac{\partial U}{\partial y}dy}{i\frac{\partial V}{\partial x}dx} = \frac{idy}{dx}.$$

In the same manner

$$\left|\frac{\partial V}{\partial y}dy\right| = \beta\frac{dy}{dx}\cos\phi = \beta\frac{dy}{dx}\frac{\frac{\partial U}{\partial x}dx}{\beta} = \frac{dy}{dx}\frac{\partial U}{\partial x}dx,$$

or in the Argand Diagram

$$\frac{i\frac{\partial V}{\partial y}dy}{\frac{\partial U}{\partial x}dx} = \frac{idy}{dx}.$$

These two results entail as the results of functionality (Cauchy's conditions).

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \qquad \frac{\partial V}{\partial y}dy = \frac{\partial U}{\partial x}dx.$$

3°. Or we might have said, since $\frac{\gamma}{dy} = \frac{\alpha}{dx}$ or $\left|\frac{\partial U}{\partial y}\right| = \left|i\frac{\partial V}{\partial x}\right|$.

Whence multiplying $i\frac{\partial V}{\partial x}$ by i revolves it into coincidence with $\frac{\partial U}{\partial y}$, that is, (see paragraph 4°) $-\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y}$.

Similarly, $\left|i\frac{\partial V}{\partial y}\right| = \left|\frac{\partial U}{\partial x}\right|$.

Whence, $i\frac{\partial V}{\partial y} = i\frac{\partial U}{\partial x}$, or $\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x}$.

4°. Since $\frac{\beta}{\delta} = \frac{dx}{dy}$, in the same way $\left|\frac{\partial W}{\partial x}\right| = \left|\frac{\partial W}{\partial y}\right|$. But $\frac{\partial W}{\partial x}$ was produced by dividing $\frac{\partial W}{\partial y}dy$ by dy. In the diagram dy is essentially negative, and remembering that division is doing to the operand what was done to the divisor to produce unity, we revolve $\frac{\partial W}{\partial y}dy$ through 180°, and change in the proper ratio, giving $\frac{\partial W}{\partial y}$ as shown in the diagram. The same remarks apply to $\frac{\partial W}{\partial x}$, but here the angle is zero.

Evidently revolving $\frac{\partial W}{\partial x}$ through 90° gives coincidence with $\frac{\partial W}{\partial y}$, or

$$i\frac{\partial W}{\partial x}=\frac{\partial W}{\partial y},$$

which is Riemann's condition of functionality.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Continued from October Number.]

Let $e_1=\frac{e^2}{2\pm e^2}$, $e_2=\frac{e_1^2}{2\pm e_1^2}$, $e_3=\frac{e_2^2}{2\pm e_2^2}$, ... , $e_n=\frac{e^2{}_{n-1}}{2\pm e^2{}_{n-1}}$.

$[(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R$,

$[(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_1$,

$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_2$,

$[(1-e_3^2\sin^2 8\theta)(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]=R_3$

. . . . . .

$[(1-e_n^2\sin^2 2^n\theta)(1-e^2{}_{n-1}\sin^2 2^{n-1}\theta)\ldots.(1-e_3^2\sin^2 8\theta)$

$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]\ \ldots R_n$.

But $R=[1\pm e^2(\sin^2\theta+\cos^2\theta)+e^4\sin^2\theta\cos^2\theta]^{\frac{1}{2}}=[1\pm e^2+\frac{1}{4}e^4\sin^2 2\theta]^{\frac{1}{2}}$

$=\frac{1}{2}[(2\pm e^2)^2-e^4\cos^2 2\theta]^{\frac{1}{2}}=\frac{1}{2}(2\pm e^2)[1-e_1^2\sin^2(\frac{1}{2}\pi-2\theta)]^{\frac{1}{2}}$.

Let $\frac{1}{2}\pi-2\theta=\varphi$. Then $d\theta=-\frac{1}{2}d\varphi$.

$$\therefore\int_0^{\frac{1}{2}\pi}Rd\theta=\tfrac{1}{4}(2\pm e^2)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(1-e_1^2\sin^2\varphi)d\varphi$$

$$=\tfrac{1}{2}(2\pm e^2)\int_0^{\frac{1}{2}\pi}(1-e_1^2\sin^2\varphi)d\varphi=\tfrac{1}{2}(2\pm e^2)E(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots\ldots(85).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R}=\frac{2}{2\pm e^2}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{(1-e_1^2\sin^2\varphi)}=\frac{2}{2\pm e^2}F(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots(86).$$

$R_1=\frac{1}{2}(2\pm e^2)[(1-e_1{}^2\sin^2 2\theta)(1-e_1{}^2\cos^2 2\theta)]^{\frac{1}{2}}$

$=\frac{1}{4}(2\pm e^2)(2-e_1{}^2)[1-e_2{}^2\sin^2(\frac{1}{2}\pi-4\theta)]^{\frac{1}{2}}$.

Let $(\frac{1}{2}\pi-4\theta)=\lambda$. $\therefore\ d\theta=-\frac{1}{4}d\lambda$.

$$\therefore \int_0^{\frac{1}{4}\pi} R_1\,d\theta=\frac{1}{4}\int_{-\frac{1}{2}3\pi}^{\frac{1}{2}\pi} R_1\,d\lambda=\frac{1}{4}\int_0^{\frac{1}{2}\pi} R_1\,d\lambda+\frac{1}{4}\int_0^{\frac{1}{2}3\pi} R_1\,d\lambda=\frac{1}{4}\int_0^{\frac{1}{2}\pi} R_1\,d\lambda$$

$$+\frac{3}{4}\int_0^{\frac{1}{2}\pi} R_1\,d\lambda=\int_0^{\frac{1}{4}\pi} R_1\,d\lambda=\frac{1}{4}(2\pm e^2)(2-e_1{}^2)\int_0^{\frac{1}{2}\pi}\frac{}{1}\sqrt{(1-e_2{}^2\sin^2\lambda)}\,d\lambda$$

$$=\frac{1}{4}(2\pm e^2)(2-e_1{}^2)E(e_2,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots\ldots\ldots(87).$$

$$\int_0^{\frac{1}{4}\pi}\frac{d\,\theta}{R_1}=4\int_0^{\frac{1}{4}\pi}\frac{d\,\lambda}{R_1}=\frac{4}{(2\pm e^2)(2-e_1{}^2)}F(e_2,\ \tfrac{1}{2}\pi)\ \ldots\ \ldots\ldots\ldots\ldots\ldots(88).$$

Similarly, $\int_0^{\frac{1}{2}\pi} R_2\,d\theta=\frac{1}{8}(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)E(e_3,\ \frac{1}{2}\pi)\ \ldots\ldots\ldots\ldots(89)$.

$$\int_0^{\frac{1}{4}\pi}\frac{d\,\theta}{R_2}=\frac{8}{(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)}F(e_3,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots\ldots\ldots(90).$$

$$\int_0^{\frac{1}{4}\pi} R_3\,d\theta=\frac{1}{2^4}(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)(2-e_3{}^2)E(e_4,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots(91).$$

$$\int_0^{\frac{1}{4}\pi}\frac{d\,\theta}{R_3}=\frac{2^4}{(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)(2-e_3{}^2)}F(e_4,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots\ldots(92).$$

. . . . . .

$$\int_0^{\frac{1}{4}\pi} R_n\,d\theta=\frac{1}{2^{n+1}}(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)\ldots\ldots(2-e_n{}^2)E(e_{n+1},\ \tfrac{1}{2}\pi)\ldots\ldots(93).$$

$$\int_0^{\frac{1}{4}\pi}\frac{d\theta}{R_n}=\frac{2^{n+1}}{(2\pm e^2)(2-e_1{}^2)(2-e_2{}^2)\ldots\ldots(2-e_n{}^2)}F(e_{n+1},\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots(94).$$

$$\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}(2m+1)}d\theta.$$

Let $\frac{1}{2}\pi-\theta=\vartheta$.

$$\therefore \int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\vartheta)^{\frac{1}{2}(2m+1)}d\vartheta$$

$$=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\vartheta)^m\frac{}{1}\sqrt{(1-e^2\sin^2\vartheta)}\,d\vartheta=M,\ \text{suppose.}$$

Let $m=0$. $\therefore\ M=\int_0^{\frac{1}{2}\pi}\frac{}{1}\sqrt{(1-e^2\sin^2\vartheta)}\,d\vartheta=E(e,\ \tfrac{1}{2}\pi)$.

Let $m=1$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^{\frac{1}{2}}/(1-e^2\sin^2\beta)d\beta$.

From (2) and (8), $M=\frac{1}{3}[2(2-e^2)E(e,\ \frac{1}{2}\pi)-(1-e^2)F(e,\ \frac{1}{2}\pi)]$ .....(95).

Let $m=2$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^2\,{}^{1}/(1-e^2\sin^2\beta)d\beta$.

From (2), (8), and (15),

$M=\frac{1}{15}[(23-23e^2-8e^4)E(e,\ \frac{1}{2}\pi)-4(2-3e^2+e^4)F(e,\ \frac{1}{2}\pi)]$ .....(96),

and so on for other values of m.

To integrate $\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}$, let $\tan\delta=\frac{1}{{}^{1}/(1-e^2)}\tan\theta$.

$$\therefore\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\delta)^{\frac{1}{2}(2m-1)}d\delta,$$

a form like preceding. When $m=0$ we get

$$\int_0^{\frac{1}{2}\pi}{}^{1}/(1-e^2\cos^2\theta)d\theta=\int_0^{\frac{1}{2}\pi}{}^{1}/(1-e^2\sin^2\theta)d\theta=E(e,\ \tfrac{1}{2}\pi).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{{}^{1}/(1-e^2\cos^2\theta)}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{{}^{1}/(1-e^2\sin^2\theta)}=F(e,\ \tfrac{1}{2}\pi).$$

By letting $\cot\gamma=\frac{1}{{}^{1}/(1-e^2)}\cot\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\gamma)^{\frac{1}{2}(2m-1)}d\gamma,$$ a previous form.

Letting $\tan^2\delta=\frac{1}{1-e^2}\tan^2\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-c\cos^2\theta)(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}$$

$$=\frac{1}{(1-c)(1-e^2)^m}\int_0^{\frac{1}{2}\pi}\frac{(1-e^2\sin^2\delta)^{m+1}d\delta}{(1+\frac{c-e^2}{1-c}\sin^2\delta)\,{}^{1}/(1-e^2\sin^2\delta)}.$$

We will now proceed to more general work.

$$\int\frac{{}^{1}/(1-e^2\sin^2\theta)d\theta}{\sin^2\theta}=-\cot\theta\,{}^{1}/(1-e^2\sin^2\theta)-e^2\int\frac{\cos^2\theta d\theta}{{}^{1}/(1-e^2\sin^2\theta)}$$

$$=(1-e^2)F(e,\ \theta)-E(e,\ \theta)-\cot\theta\,{}^{1}/(1-e^2\sin^2\theta)=u_1\ldots.(97).$$

$$\int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta} = \tan\theta\sqrt{(1-e^2\sin^2\theta)} + e^2\int \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$= F(e,\ \theta) - E(e,\ \theta) + \tan\theta\sqrt{(1-e^2\sin^2\theta)} = R_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (98).$$

$$\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} = e^2\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + \int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^2\theta}$$

$$= F(e,\ \theta) - E(e,\ \theta) - \cot\theta\sqrt{(1-e^2\sin^2\theta)} = V_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (99).$$

$$\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{1-e^2}\int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta} - \frac{e^2}{1-e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$= F(e,\ \theta) - \frac{1}{1-e^2}E(e,\ \theta) + \frac{\tan\theta}{1-e^2}\sqrt{(1-e^2\sin^2\theta)} = S_1 \quad \ldots\ldots\ldots (100).$$

$$\int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta} = -\frac{\cos\theta}{3}\left(\frac{1}{\sin^3\theta} + \frac{2}{\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)} + \frac{e^2}{3}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$- \frac{e^2}{3}\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} - \frac{2e^2}{3}\int \frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = \frac{2(1-e^2)}{3}F(e,\ \theta) -$$

$$\frac{(2-e^2)}{3}E(e,\ \theta) - \frac{(2-e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)} - \frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)} = U_2 \ldots (101).$$

$$\int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta} = \frac{\sin\theta}{3}\left(\frac{1}{\cos^3\theta} + \frac{2}{\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+ \frac{e^2}{3}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} - \frac{e^3}{3}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + \frac{2e^2}{3}\int \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$= \frac{2}{3}F(e,\ \theta) - \frac{(2-e^2)}{3(1-e^2)}E(e,\ \theta) + \frac{(2-e^2)}{3(1-e^2)}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+ \frac{\sin\theta}{3\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)} = R_2 \quad \ldots\ldots\ldots\ldots (102).$$

$$\int \frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}} = e^2\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} + \int \frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$= \frac{(2+e^2)}{3}F(e,\ \theta) - \frac{2(1+e^2)}{3}E(e,\ \theta) - \frac{2(1+e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$- \frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)} = V_2 \quad \ldots\ldots\ldots\ldots (103).$$

$$\int \frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{1}{1-e^2}\int \frac{\sqrt{(1-e^2\sin^2\theta)}\,d\theta}{\cos^4\theta} - \frac{e^2}{1-e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{(2-3e^2)}{3(1-e^2)}F(e,\ \theta)-\frac{2(1-2e^2)}{3(1-e^2)^2}E(e,\ \theta)+\frac{2(1-2e^2)}{3(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{3(1-e^2)\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_2 \quad \ldots\ldots\ldots(104).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}=-\frac{\cos\theta}{5}\left(\frac{1}{\sin^5\theta}+\frac{4}{3\sin^3\theta}+\frac{8}{3\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{e^2}{5}\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{15}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{8e^2}{15}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-7e^2-e^4)}{15}F(e,\ \theta)-\frac{(8-3e^2-2e^4)}{15}E(e,\ \theta)$$

$$-\frac{(8-3e^2-2e^4)}{15}\cot\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{(4-e^2)}{15}\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}=U_3 \quad \ldots\ldots\ldots\ldots(105).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}=\frac{\sin\theta}{5}\left(\frac{1}{\cos^5\theta}+\frac{4}{3\cos^3\theta}+\frac{8}{3\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{e^2}{5}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{e^2}{15}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$+\frac{8e^2}{15}\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-9e^2)}{15(1-e^2)}F(e,\ \theta)-\frac{(8-13e^2+3e^4)}{15(1-e^2)^2}E(e,\ \theta)$$

$$+\frac{8-13e^2+3e^4}{15(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}+\frac{4-3e^2}{15(1-e^2)}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}=R_3 \quad \ldots\ldots\ldots\ldots(106).$$

$$\int\frac{d\theta}{\sin^6\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}$$

$$=\frac{(8+3e^2+4e^4)}{15}F(e,\ \theta)-\frac{(8+7e^2+8e^4)}{15}E(e,\ \theta)-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{(8+7e^2+8e^4)}{15}\tan\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{4(1+e^2)}{15}\cdot\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=V_3 \quad \ldots(107).$$

[To be Concluded.]

GAUSS AND THE NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

Carl Friedrich Gauss Werke, Band VII. Goettingen. 1900. 4to. Pp. 458.

We are so accustomed to the German professor who does, we hardly expect the German professor who does not. Such, however, was Schering of Goettingen, who so long held possession of the papers left by Gauss.

Schering had planned and promised to publish a supplementary volume, but never did, and only left behind him at his death certain preparatory attempts thereto, consisting chiefly of excerpts copied from the manuscripts and letters left by Gauss. Meantime these papers for all these years were kept secret and even the learned denied all access to them.

Schering dead, his work has been quickly and ably done, and here we have a stately quarto of matter supplemental to the first three volumes, and to the fourth volume with the exception of the geodetic part.

Of chief interest for us is the geometric portion, pages 159—452, edited by just the right man, Professor Staeckel of Kiel.

One of the very greatest discoveries in mathematics since ever the world began is, beyond peradventure, the non-Euclidean geometry.

By whom was this given to the world in print? By a Hungarian, John Bolyai, who made the discovery in 1823, and by a Russian, Lobachévski, who had made the discovery by 1826. Were either of these men prompted, helped, or incited by Gauss, or by any suggestion emanating from Gauss? No, quite the contrary.

Our warrant for saying this with final and overwhelming authority is this very eighth volume of Gauss's works, just now at last put in evidence, published to the world.

The geometric part opens, page 159, with Gauss's letter of 1799 to Bolyai Farkas the father of John (Bolyai János), which I gave years ago in my Bolyai as demonstrative evidence that in 1799 Gauss was still trying to prove Euclid's the only non-contradictory system of geometry, and also the system of objective space. The first is false; the second can never be proven.

But both these friends kept right on working away at this impossibility, and the more hot-headed of the two, Farkas, finally thought he had succeeded with it, and in 1804 sent to Gauss his "Goettingen Theory of Parallels." Gauss's judgment on this is the next thing given (pages 160—2). He shows the weak spot. "Could you *prove*, that $dkc=ckf=fkg$, etc., then were the thing perfect. However, this theorem is indeed true, only difficult, with already presupposing the theory of parallels, to prove rigorously." Thus in 1804 instead of having or giving any light, Gauss throws his friend into despair by intimating that the link missing in his labored attempt is true enough but difficult to prove without *petitio principii*.

Of course we now know it is *impossible* to prove. Anything is impossible to prove which is the equivalent of the parallel-postulate. Yet both the friends continue their strivings after this impossibility.

In this very letter Gauss says: "I have indeed yet ever the hope, that those rocks sometime, and indeed before my end, will allow a through passage."

Farkas in 1808, December 27, writes to Gauss: "Oft thought I, gladly would I, as Jacob for Rachel serve, in order to know the parallels founded, even if by another.

Now just as I thought it out on Christmas night, while the Catholics were celebrating the birth of the Saviour in the neighboring church, yesterday wrote it down, I send it to you enclosed herewith. To-morrow must I journey out to my land, have no time to revise, neglect I it now, may be a year is lost, or indeed find I the fault, and send it not, as has already happened with hundreds, which I as I found them took for genuine. Yet it did not come to writing those down, probably because they were too long, too difficult, too artificial, but the present I wrote off at once. As soon as you can, write me your real judgment."

This letter Gauss never answered, and never wrote again until 1832, a quarter of a century later, when the non-Euclidean geometry had been published by both Lobachévski and Bolyai János.

This settles now forever all question of Gauss having been of the slightest or remotest help or aid to young János, who in 1823 announced to his father Farkas in a letter still extant, which I saw at the Reformed College in Maros-Vásárhely, where Farkas was professor of mathematics, his discovery of the non-Euclidean geometry as something undreamed of in the world before.

This immortal letter, a charming and glorious outpouring of pure young genius, speaks as follows:

"Temesvár, 3 Nov. 1823.

My dear and good father. I have so much to write of my new creations, that it is at the moment impossible for me to enter into great detail, so I write you only on a quarter of a sheet. I await your answer to my letter of two sheets; and perhaps I would not have written you before receiving it, if I had not wished to address to you the letter I am writing to the Baroness, which letter I pray you to send her.

First of all I reply to you in regard to the binomial.

* * * * * * * * *

Now to something else, so far as space permits. I intend to write, as soon as I have put it into order, and when possible to publish, a work on parallels. At this moment it is not yet finished, but the way I have hit upon promises me with certainty the attainment of the goal, if it in general is attainable. It is not yet attained, but I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it. Now I cannot say more of them, only so much: *that from nothing I have created a wholly new world.* All that I have hitherto sent you compares to this only as a house of cards to a castle.

P. S. I dare to judge absolutely and with conviction of these works of my spirit before you, my father; I do not fear from you any false interpretation (that certainly I would not merit), which signifies that, in certain regards, I consider you as a second self."

In his Autobiography János says: "First in the year 1823 did I completely penetrate through the problem according to its essential nature, though also afterward further completions came thereto. I communicated in the year 1825 to my former teacher, Herrn Johann Walter von Eckwehr (later imperial-royal general), a written paper, which is still in his hands. On the prompting of my father I translated my paper into Latin, in which it appeared as *Appendix* to the *Tentamen* in 1832."

So much for Bolyai.

The equally complete freedom of Lobachévski from the slightest idea that Gauss had ever meditated anything different from the rest of the world on the parallels I showed in "Science," Vol. IX, No. 232, page 813—817.

Passing on to the next section, pages 163-4, in the new volume of Gauss, we find it important as showing that in 1805 Gauss was still a baby on this subject. It is an erroneous pseudo-proof of the impossibility of what in 1733 Saccheri had called "hypothesis anguli obtusi." To be sure Saccheri himself thought he had proven this hypothesis inadmissable, so that Gauss blundered in good company; but his pupil Riemann in 1854 showed that this hypothesis gives a beautiful non-Euclidean geometry, a new universal space, now justly called the space of Riemann.

Passing on, we find that in 1808, Schumacher writes: "Gauss has led back the theory of parallels to this, that if the accepted theory were not true, there must be a constant *a priori* line given in length, which is absurd. Yet he himself considers this work still not conclusive."

Again, with the date April 27, 1813, we read: "In the theory of parallels we are even now not farther than Euclid was. This is the *partie honteuse* (shameful part) of mathematics, which soon or late must receive a wholly different form."

Thus in 1813 there is still no light.

In April, 1816, Wachter on a visit to Goettingen had a conversation with Gauss whose subject was what he calls the anti-Euclidean geometry. On December 12, 1816, he writes to Gauss a letter which shows that this anti-Euclidean geometry, as he understands it, far from being the non-Euclidean geometry of Lobachévski and Bolyai János, was a monstrous conglomerate blunder.

The letter as here given by Staeckel, pages 175—176, is as follows:

* * * "Consequently the anti-Euclidean or your geometry would be true. However the *constant* in it remains undetermined: why? may perhaps be made comprehensible by the following. * * * The result of the foregoing may consequently be so expressed:

The Euclidean geometry is false; but nevertheless the true geometry must begin with the same eleventh Euclidean axiom or with the assumption of

lines and surfaces which have the property presumed in that axiom. Only instead of the straight line and plane are to be put the great circle of that sphere described with infinite radius together with its surface. From this comes indeed the one inconvenience, that the parts of this surface are merely symmetric, not, as with the plane, congruent; or that the radius out on the one side is infinite, on the other imaginary. Only it is clear how that inconvenience is again overbalanced by many other advantages which the construction on a spherical surface offers: *so that probably also then even*, if the Euclidean geometry were true, the necessity no longer indeed exists, to consider the plane as an infinite spherical surface, though still the fruitfulness of this view might recommend it.

Only, as I thought through all this, as I had already fully settled myself about the result, in part since I believed I had recognized the ground (la métaphysique) of that indeterminateness necessarily inherent in geometry,—also even the complete indecision in this matter, then, if that proof against the Euclidean geometry, as I could not expect, were not to be considered as stringent—; in part, while yet not to consider as lost all the many previous researches in plane geometry: but to be used with few modifications, and if still also the theorems of solid geometry and mechanics might have approximate validity, at least to a quite wide limit, which perhaps yet could be more nearly determined; I found this evening—just while busied with an attempt to find an entrance to your transcendental trigonometry, and while I could not find in the plane sufficing, determinate functions thereto, going on to space constructions, to my no small delight the following *demonstration for the Euclidean parallel-theory.* * * * Just in the idea to conclude I remark still, that the above proof for the Euclidean parallel-theory is fallacious. * * * Consequently has here also the hope vanished, to come to a fully decided result, and I must content myself again with the above cited. Withal I believe I have made upon that way at least a step toward your transcendental trigonometry, since I, with aid of the spherical trigonometry, can give the ratios of all constants, at least *by construction of the right-angled triangle.* I yet lack the actual reckoning of the base of an isosceles triangle from the side, to which I will seek to go from the equilateral triangle."

If Gauss's transcendental trigonomety were as sad a hodge-podge as the anti-Euclidean geometry here explained by Wachter, it is fortunate that nothing was ever given about it but its name. *Requiescat in pace.*

Yet Gauss writes, April 28, 1817:

"Wachter has printed a little piece on the foundations of geometry. Though Wachter has penetrated farther into the essence of the matter than his predecessors, yet is his proof not more valid than all others."

We come now to an immortal epoch, that of the discovery of the real non-Euclidean geometry by Schweikart, and his publication of it under the name of Astralgeometry.

On the twenty-fifth of January, 1819, Gerling writes to Gauss:

"Apropos of parallel-theory I must tell you something, and execute a commission. I learned last year, that my colleague Schweikart (prof. juris,

now Prorector) formerly occupied himself much with mathematics and particularly also had written on parallels. So I asked him to lend me his book. While he promised me this, he said to me, that now indeed he perceived how errors were present in his book (1808) (he had, for example, used quadrilaterals with equal angles as a primary idea), however that he had not ceased to occupy himself with the matter, and was now about convinced, that without some datum the Euclidean postulate could not be proved, also that it was not improbable to him, that our geometry is only a chapter of a more general geometry.

Then I told him how you some years ago had openly said, that since Euclid's time we had not in this really progressed; yes, that you had often told me, how you through manifold occupation with this matter had not attained to the proof of the absurdity of such a supposition.—Then when he sent me the book asked for, the enclosed paper accompanied it, and shortly after (end of December) he asked me orally, when convenient to enclose to you this paper of his, and to ask you in his name to let him know when convenient your judgment on these ideas of his.

The book itself has, apart from all else, the advantage that it contains a copious bibliography of the subject; which he also, as he tells me, has not ceased still further to add to."

Now comes, pages 180—181, the precious enclosure, dated Marburg, December, 1818, which, though so brief, may fairly be considered the first *published* [not printed] treatise on non-Euclidean geometry. It is a pleasure to give this here in English for the first time.

THE NON-EUCLIDEAN GEOMETRY OF 1818.

BY SCHWEIKART.

"There is a two-fold geometry,—a geometry in the narrower sense—the Euclidean; and an astral science of magnitude.

The triangles of the latter have the peculiarity, that the sum of tha three angles is not equal to two right angles.

This presumed, it can be most rigorously proven:

(*a*) That the sum of the three angles in the triangle is *less* than two right angles;

(*b*) that this sum becomes ever smaller, the more content the triangle encloses;

(*c*) That the altitude of an isosceles right-angled triangle indeed ever increases, the more one lengthens the side, that it however cannot surpass a certain line, which I call the *constant*.

Squares have consequently the following form:

Is this constant *for us* half the earth's axis (as a consequence of which each line drawn in the universe from one fixed star to another, which are ninety degrees apart from one another, would be a tangent of the earth-sphere), so is it in relation to the spaces occuring in daily life infinitely great.

The Euclidean geometry holds good only under the presupposition, that

the constant is infinitely great. Only then is it true, that the three angles of every triangle are equal to two right angles; also this can be easily proven if one takes as given the proposition, that the constant is infinitely great."

Such is the brief declaration of independence of this hero.

Nor was Schweikart's courage and independence without farther issue. Under his direct influence his own nephew Taurinus developed the real non-Euclidean trigonometry and published it in 1825 with successful application to a number of problems.

Moreover this teaching of Schweikart's made converts in high places. In the letter of Bessell to Gauss of February 10, 1829 (page 201) he says:

"Through that which Lambert said, and what Schweikart disclosed orally, it has become clear to me, that our geometry is incomplete, and should receive a correction, which is hypothetical and, if the sum of the angles of the plane triangle is equal to a hundred and eighty degrees, vanishes.

That were the *true* geometry, the Euclidean the *practical*, at least for figures on the earth."

The complete originality and independence of Schweikart and of Lobachévski is recognized as a matter of course in the correspondence between Gauss and Gerling, who writes, page 238, "The Russian steppes seem therefore indeed a proper soil for these speculations, for Schweikart (now in Koenigsberg) invented his 'Astral-Geometry' while he was in Charkow."

This fixes the date of the first conscious creation and naming of the non-Euclidean geometry as between 1812 and 1816.

Gauss adopts and uses for himself this first name, Astralgeometry [1832, page 226; 1841, page 232].

At length the true prince comes. On February 14, 1832, Gauss receives the profound treatise of the young Bolyai János, the most marvellous two dozen pages in the history of thought. Under the first impression Gauss writes privately to his pupil and friend Gerling of the ideas and results as "mit grosser eleganz entwickelt." He even says "I hold this young geometer von Bolyai to be a genius of the first magnitude."

Now was Gauss's chance to connect himself honorably with the non-Euclidean geometry, already independently discovered by Schweikart, by Lobachévski, by Bolyai János.

Of two utterly worthless theories of parallels Gauss had already given extended notices in the Goettingische gelehrte Anzeigen (this volume pages 170—174, and 183—185).

To this marvel of János, Gauss vouchsafed never one printed word.

As Staeckel gently remarks, this certainly contributed thereto, that the worth of this mathematical gem was first recognized when John had long since finished his earthly career.

The 15th of December, 1902, will be the centenary of the birth of Bolyai János. Should not the learned world endeavor to arouse the Magyars to honor Hungary by honoring then this truest genius her son?

Austin, Texas.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the October Number.]

CHAPTER X.

146. Combinatory multiplication gives us a very neat solution of sets of simultaneous equations. Let n equations of the first degree containing n unknowns be given to find the value of the unknowns. Let the n equations be

$$\alpha_{11}x_1+\alpha_{21}x_2+\ldots.+\alpha_{n1}x_n=\beta_1$$
$$\alpha_{12}x_1+\alpha_{22}x_2+\ldots.+\alpha_{n2}x_n=\beta_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{1n}x_1+\alpha_{2n}x_2+\ldots.+\alpha_{nn}x_n=\beta_n$$

We multiply the first of these equations through by e_1, the second through by e_2, and so on, and the last by e_n, (where $[e_1e_2\ldots.e_n]=1$), and add the resulting equations. Then if

$$\alpha_{11}e_1+\alpha_{12}e_2+\ldots.+\alpha_{1n}e_n=a_1$$
$$\alpha_{21}e_1+\alpha_{22}e_2+\ldots.+\alpha_{2n}e_n=a_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{n1}e_1+\alpha_{n2}e_2+\ldots.+\alpha_{nn}e_n=a_n$$
$$\beta_1e_1+\beta_2e_2+\ldots\ldots+\beta_ne_n=b,$$

we have for the sum of the products referred to above, the equation

$$x_1a_1+x_2a_2+\ldots.+x_na_n=b.$$

By 20 this one equation replaces or transforms the set originally given us. Now in order to find x_1, multiply the last equation by $[a_2a_3\ldots.a_n]$. This gives (52, 43)

$$x_1[a_1a_2\ldots.a_n]=[ba_2a_3\ldots.a_n]$$
$$\text{or, } x=\frac{[ba_2a_3\ldots.a_n]}{[a_1a_2\ldots.a_n]}$$

Replacing the a's and b by their values, by 45 we have the usual expression for x_1 in terms of the coefficients.

CHAPTER XI.

APPLICATION TO TRIGONOMETRY.

147. DEFINITION.—The angle between two quantities is that angle ($<\pi$) whose cosine equals the inner product of the two quantities divided by the product of their scalar coefficients. Thus

$$\cos \angle ab = [a \mid b] \div \alpha\beta$$

where a and b are two quantities and α and β are their numerical values (123).

Again, if $a, b, c \ldots$ are quantities of the first order, $\alpha, \beta, \gamma \ldots$ are their respective numerical values. $\sin(a\ b\ c \ldots)$ is that numerical quantity which equals

$$\frac{[a\ b\ c \ldots]}{\alpha\beta\gamma \ \ldots}$$

and is not negative. Thus (123) $\sin^2(abc) = \dfrac{[a\ b\ c \ldots]^2}{\alpha^2\beta^2\gamma^2 \ldots}$.

148. *If a and b are quantities of the first order,* $\sin(ab) = \sin \angle ab$.

PROOF.— $\sin^2(ab) = \dfrac{[a\,b]^2}{\alpha^2\beta^2} = \dfrac{a^2b^2 - [a \mid b]^2}{\alpha^2\beta^2}$ (144)

$$= \frac{\alpha^2\beta^2 - [a \mid b]^2}{\alpha^2\beta^2} \ (123) = 1 - \frac{[a \mid b]^2}{\alpha^2\beta^2} \ (123)$$

$$= 1 - \cos^2 \angle ab = \sin^2 \angle ab \ (147).$$

Then if $\sin(ab)$ is never negative and $\angle ab < \pi$, $\sin(ab) = \sin \angle ab$.

149. *If $\alpha, \beta, \gamma, \delta$ are the numerical values of a, b, c, d, by* 147 *and* 148,

$$[ab \mid cd] = \alpha\beta\gamma\delta \sin \angle ab \sin \angle cd \cos \angle (ab.cd).$$

150. *The normal projection of A on a quantity B of the same order is numerically equal to $A\cos \angle AB$.*

PROOF.—If A' is the normal projection of A on B (134)

$$A' = \frac{[A \mid B]B}{\beta^2} = \frac{\alpha\beta \cos \angle AB.B}{\beta^2} \ (119, 144)$$

$$= \alpha \cos \angle AB \,.\, \frac{B}{\beta} = \text{(numerically) } A \cos \angle AB.$$

151. The two expressions $[a \mid b]$ and $[ab]$, where a and b are vectors, play a very important part in mathematics. They occur yoked together in quaternions and apart in the *Ausdehnungslehre*, typifying the two products, the inner and outer. Numerically, as we have just seen, $[a \mid b]$ is the *projection* of either vector on the other multiplied by the tensor of the other ; $[ab]$, on the other

hand, is the area of the parallelogram whose adjacent sides are a and b, or, when the tensor of one vector is unity, it is equal numerically to the *perpendicular* from the extremity of the other vector on the first, when they go out from the same origin, or, when both tensors are unity, it is equal numerically to the sine of the angle between the given vectors (148).

152. *If a, b, c, ... are normal to each other and k is any quantity numerically derived from them, we have*

$$\frac{k}{\varkappa}=\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.$$

Proof.—Let $k=xa+yb+\ldots.$ Then to find x multiply each member by $\mid a$. There results, since $[b \mid a]=0$, etc., $[k \mid a]=x[a \mid a]$. Finding the value of $y\ldots.$ in the same way and substituting we get the equation as given above.

153. If a, b, $c\ldots.$ are normal to one another and k and l are two quantities numerically derivable from a, b, ...

$$\cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

Proof.—From 147, we have

$$\cos\angle kl=\frac{[k \mid l]}{\varkappa\lambda}=\left[\frac{k}{\varkappa}\middle|\frac{l}{\lambda}\right]\equiv$$

$$\left[\left(\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.\right)\middle|\left(\frac{a}{\alpha}\cos\angle al+\frac{b}{\beta}\cos\angle bl+\ldots.\right)\right] \quad (152)$$

$$=\frac{a^2}{\alpha^2}\cos\angle ak\cos\angle al+\frac{b^2}{\beta^2}\cos\angle bk\cos\angle bl+\ldots.$$

$\therefore\ \cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$

154. If a, b, c, ... are normal to each other and k is numerically derivable from them, we have by putting $l=k$ in 153

$$1=\cos^2\angle ak+\cos^2\angle bk+\ldots.$$

155. If a, b, c, ... are normal to each other, and k and l two quantities numerically derivable from them are normal to each other, 153 gives

$$0=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

156. Writing in the formula of 144 a for p_1, b for p_2, c for q_1, d for q_2 gives

$$\sin\angle ab\sin\angle cd\cos\angle(ab.cd)=\cos\angle ac\cos\angle bd-\cos\angle bc\cos\angle ad.$$

In this formula if c and d are replaced respectively by a and c there results

$$\sin\angle ab\sin\angle ac\cos\angle(ab.ac)=\cos\angle bc-\cos\angle ba\cos\angle ac,$$

a familiar formula of spherical trigonometry.

157. The last formula of 145, by substituting a, b, c for p_1, p_2, p_3, and a, b, c for q_1, q_2, q_3 gives (147)

$$\sin^2(abc)=1-\cos^2\angle bc-\cos^2\angle ca-\cos^2\angle ab+2\cos\angle bc\cos\angle ca\cos\angle ab.$$

158. *The formula* $(a+b)^2=a^2+2[a\mid b]+b^2=\alpha^2+2\alpha\beta\cos\angle ab+b^2$ gives the familiar extension of the Pythagorean proposition.

159. Let a, b, c be plane segments whose sum is the fourth face of a tetraedron of which they are the other three (75). Then

$$\begin{aligned}(a+b+c)^2&=a^2+b^2+c^2+2[b\mid c]+2[c\mid a]+2[a\mid b]\\&=\alpha^2+\beta^2+\gamma^2+2\beta\gamma\cos\angle bc+2\alpha\gamma\cos\angle ca+2\alpha\beta\cos\angle ab,\end{aligned}$$

which is the extension of the preceding result to space.

In words:—The square of the base of any tetraedron is equal to the sum of the squares of the lateral faces dimished by twice the products of each pair of lateral faces times the cosine of the diedral angle between them.

CHAPTER XII.

APPLICATION TO ANALYTIC GEOMETRY.

160. Let p_1, p_2, p_3 represent three unit points, and suppose their product is unity (57). Then

$$[p_1p_2p_3]=[p_1(p_2p_3+p_3p_1+p_1p_2)]=1. \quad (43)$$

But if p denote any other unit point in the plane of $[p_1p_2p_3]$, by 94 we may replace p_1 by p in this product, getting

$$[p(p_2p_3+p_3p_1+p_1p_2)]=[p\mid(p_1+p_2+p_3)]=1. \quad (57, 58)$$

Let p_s denote a unit point which is the mean of the reference points. Then $p_1+p_2+p_3=3p_s$ (81). Substituting this value of $p_1+p_2+p_3$ in the equation above, we have

$$3[p\mid p_s]=1, \text{ or in solid space, } 4[p\mid p_s]=1.$$

161. *The equation* $p=xp_1+yp_2+zp_3$ *represents a straight line, provided* x, y, z *satisfy a linear equation, as* $ax+by+cz=0$.

To see this let us eliminate z. Then

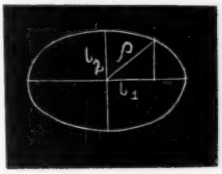
$$p=\frac{1}{c}\{x(cp_1-ap_3)+y(cp_2-bp_3)\}.$$

Thus, by 80, p lies on the right line through cp_1-ap_3 and cp_2-bp_3.

162. The equation $[p\,p_1p_2]=0$, in which p_1 and p_2 are constants and p is a variable is the equation of a straight line (94).

The equation $[pL]=0$, *where* p *is a point and* L *a line is the point equation of a straight line if* p *is variable and* L *is constant, and the line equation of the point* p *if* L *is variable and* p *is constant.*

163. *The Cartesian equations of the central conics, the ellipse and the hyperbola in the inner product notation* (151) *are*



$$\left(\frac{\rho\,|\,\iota_1}{a}\right)^2\pm\left(\frac{\rho\,|\,\iota_2}{b}\right)^2=1,$$

where ι_1 and ι_2 are unit vectors along the major and minor axes and ρ is the radius vector from the center to any point. Suppose we set

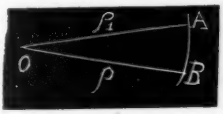
$$\frac{\rho\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho\,|\,\iota_2}{b^2}\iota_2\equiv\phi\rho.$$

Then the equation for the central conics reduces to

$$\rho\,|\,\phi\rho=1.$$

164. DIFFERENTIATION.—Let ρ be a radius vector from an origin O to a curve AB. Then if ρ be made to approach indefinitely close to ρ_1, we have

$$\text{Limit}\frac{\rho-\rho_1}{AB}=\frac{d\rho}{ds}=a\ unit\ vector$$



in the direction of the tangent at A. This is taken to be the meaning of $\dfrac{d\rho}{ds}$ no matter whether ρ be a *vector* from the origin O, or a *point* moving from B to A on the curve AB.

165. The function ϕ (163) possesses the property that $\rho\,|\,\phi\rho_1=\rho_1\,|\,\phi\rho$. Thus

$$\rho\,|\,\phi\rho_1\equiv\rho\,|\left(\frac{\rho_1\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho_1\,|\,\iota_2}{b^2}i_2\right)=\rho_1\,|\left(\frac{\rho\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho\,|\,\iota_2}{b^2}\iota_2\right).$$

166. Differentiating the equation $\rho\,|\,\phi\rho=1$ (163), we get

$$d\rho\,|\,\phi\rho+\rho\,|\,\phi d\rho=2d\rho\,|\,\phi\rho=0. \quad (165)$$

Now if $d\rho$ is parallel to the tangent at the extremity of ρ, $\phi\rho$ is parallel to the normal (124).

If ρ_t and ρ_n be vectors to any point of the tangent and normal, respectively, and ρ_1 that to the point of contact, the equation of the tangent may be written $[(\rho_t-\rho_1)\,|\,\phi\rho_1]=0$, or $[\rho_t\,|\,\phi\rho_1]=1$, and that of the normal $[(\rho_n-\rho_1)\phi\rho_1]=0$.

The p's may also be thought of as representing points.

167. Let p_1, p_2, p_3 denote the vertices of a reference triangle whose sides are of unit length and p any point in their plane. Then $|p_1 = p_2p_3$, $|p_2 = p_3p_1$, $|p_3 = p_1p_2$, and $p|p_1$, $p|p_2$, $p|p_3$ are proportional to the perpendiculars from p on the several sides of the triangle (94).

We shall consider only homogeneous equations. For, if any equation should not be homogeneous in p, all that is necessary to make it such is to introduce the factor $1 = 3p|p_s$ (160). Now the most general form of the equation of the second degree in trilinear coördinates is

$$a[p|p_1]^2 + b[p|p_2]^2 + c[p|p_3]^2 + 2d[p|p_2][p|p_3] + 2e[p|p_3][p|p_1] + 2f[p|p_1][p|p_2] = 0.$$

Let $[(ap_1 + fp_2 + ep_3)p|p_1] + [(fp_1 + bp_2 + dp_3)p|p_2] + [(ep_1 + dp_2 + cp_3)p|p_3] \equiv \phi p.$

When this value of ϕp is substituted in the preceding equation it reduces to $p|\phi p = 0$. Hence $p|\phi p = 0$ is the equation for all quadric curves whether central or non-central. Had quadriplaner coördinates been employed and the corresponding expressions constructed, an equation would have resulted representing any and all quadric surfaces. The same method may be used in getting the equation of the quadric in n-dimensional space.

REMARK.—The introduction of the ϕ function from Hamilton into the *Ausdehnungslehre* is due to Professor Hyde. (*Directional Calculus*, page 103). He shows that point analysis gives a means of changing the ordinary Cartesian equations into equations analogous to those of trilinear coördinates and then of generalizing the application of the equation $p|\phi p = 0$ to include the case of quadrics, central and non-central.

[To be Concluded.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

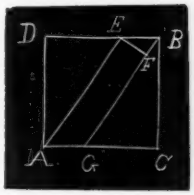
ARITHMETIC.

132. Proposed by WILLIAM SYMMONDS, A.M., Professor of Mathematics, Santa Rosa College, Sebastopol, Cal.

A road 60 feet wide crosses a square acre of land. The west line of the road passes through the southwest corner of the land, while the east line of the former passes through the northeast corner of the latter. What fraction of the land is included in the road?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; D. G. DORRANCE, Jr., Camden, N. Y.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $ACBD$=the square acre, $AGBE$=the road, and EF perpendicular to BG, =60 feet.



Put $a=AC=BC$, and $b=EF$.

Let $x=BE$, and $y=BG$.

Then $a-x=GC$, ax=area of road, and x/a=fractional part of square acre included in the road.

From the similar right triangles EFB and BCG, we have $x:y=b:a$; whence $y=ax/b$.

Also from right triangle BCG, $y=\sqrt{[a^2+(a-x)^2]}$.

$$\therefore \frac{a^2x^2}{b^2}=2a^2-2ax+x^2.$$

Whence $x=\dfrac{ab[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$, and $\dfrac{x}{a}=\dfrac{b[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$.

Substituting the numerical values, $b=60$ and $a^2=43560$ [=the area of an acre in square feet], we obtain $x/a=.344$.

QUERY. When $a=b$, what is the value of $\dfrac{ab[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$? GRUBER.

ANSWER. By differentiating both numerator and denominator with respect to b and then reducing we find the value of the expression to be equal to a, for either the + or − sign. It may also be shown as follows:

$$\frac{ab[\pm\sqrt{(2a^2-b^2)}-b]}{a^2-b^2}=\frac{ab[\pm\sqrt{(2a^2-b^2)}-b]}{\frac{1}{2}(2a^2-b^2-b^2)}$$

$$=\frac{2ab[\pm\sqrt{(2a^2-b^2)}-b]}{[\sqrt{(2a^2-b^2)}+b]\,[\sqrt{(2a^2-b^2)}-b]}=\frac{2ab}{\sqrt{(2a^2-b^2)}+b}\text{ or }-\frac{2ab}{\sqrt{(2a^2-b^2)}-b}$$

$$=a\text{ or }-\infty.$$

These values might have been found by making the assumption that $a=b$ in the equation from which the expression arose.

If, however, we write the denominator of the expression for the roots $2a^2-b^2-a^2=[\sqrt{(2a^2-b^2)}+a][\sqrt{(2a^2-b^2)}-a]$, and then divide $+\sqrt{(2a^2-b^2)}-b$ by $[\sqrt{(2a^2-b^2)}-a]$ we get

$$1+\frac{a-b}{\sqrt{(2a^2-b^2)}}+\frac{a(a-b)}{(2a^2-b^2)}+\frac{a^2(a-b)}{(2a^2-b^2)^{\frac{3}{2}}}+\text{etc.},$$

and the value of the root is

$$\frac{ab}{[\sqrt{(2a^2-b^2)}+a]}\left(1+\frac{a-b}{(2a^2-b^2)^{\frac{1}{2}}}+\frac{a(a-b)}{(2a^2-b^2)}+\frac{a^2(a-b)}{(2a^2-b^2)^{\frac{3}{2}}}+\text{etc.}\right)$$

While each of these terms after the first, approach 0 as $a \doteq b$, making it appear that from this view one root is $\frac{1}{2}a$ instead of a, as found above; yet by writing the series as follows:

$$1+(a-b)\left[\frac{a}{\sqrt{(2a^2-b^2)}}+\frac{a^2}{(2a^2-b^2)}+\frac{a^3}{(2a^2-b^2)^{\frac{3}{2}}}+\text{ etc.}\right],$$

it is seen that this factor takes the form, in the limit, $1+0\times\infty$; and, therefore, this method is no more capable of yielding a determinate result than is the original expression. EDITOR F.

ALGEBRA.

108. Proposed by GEORGE LILLEY, Ph.D., LL.D., Professor of Mathematics, State University, Eugene, Or.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent. annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the endorsement on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$.

[This problem is the same as No. 86, Miscellaneous. See the solutions in that department.]

109. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

$\frac{1}{\sqrt{(x+1)}}+\frac{1}{\sqrt{(x-1)}}=\frac{1}{\sqrt{(x^2-1)}}$; find value of x satisfying the equation.

Solution by the PROPOSER.

This problem was proposed for the purpose of explaining the singular fact (singular to those who do not possess more than a mechanical knowledge of algebra) that the value of the unknown does not satisfy the original equation.

By transposing and factoring, the original equation may be written

$$\frac{1}{\sqrt{(x^2-1)}}[\sqrt{(x-1)}+\sqrt{(x+1)}-1]=0,$$

which is equivalent to the system of equations,

$$\left\{\begin{array}{l}\frac{1}{\sqrt{(x^2-1)}}=0 \ldots\ldots\ldots\ldots\ldots A. \\ \sqrt{(x-1)}+\sqrt{(x+1)}-1=0 \ldots\ldots\ldots B.\end{array}\right\}$$

The solution of A gives $x=\infty$, which value satisfies the original equation.

From B we have $\sqrt{(x+1)}=1-\sqrt{(x-1)}$. Squaring both sides, transposing and combining, we have $1=-2\sqrt{(x-1)}$.

From this last, by squaring, we obtain $1=4(x-1)$, from which we find,

$x=\frac{5}{4}$. This value of x does not satisfy B, neither does it satisfy the original equation.

By studying the two operations of squaring, in the solution of B, it will be seen that extraneous equations were introduced. The squaring the first time was equivalent to multiplying B by $\sqrt{(x-1)}+\sqrt{(x+1)}+1$, that is, the equation resulting from squaring the first time is equivalent to

$$[\sqrt{(x-1)}+\sqrt{(x+1)}+1][\sqrt{(x-1)}+\sqrt{(x+1)}-1]=0\ldots.(1),$$

and the second operation of squaring which gives the equation $1=4(x-1)$ is equivalent to multiplying (1) by $[\sqrt{(x-1)}-\sqrt{(x+1)}+1][\sqrt{(x-1)}-\sqrt{(x+1)}-1]$, that is, the equation $1=4(x-1)$ is equivalent to the equation

$$[\sqrt{(x-1)}+\sqrt{(x+1)}+1][\sqrt{(x-1)}+\sqrt{(x+1)}-1]$$
$$[\sqrt{(x-1)}-\sqrt{(x+1)}+1][\sqrt{(x-1)}-\sqrt{(x+1)}-1]=0.$$

This equation, therefore, is equivalent to the system of equations

$$\left\{\begin{array}{l}\sqrt{(x-1)}+\sqrt{(x+1)}+1=0=P\\ \sqrt{(x-1)}+\sqrt{(x+1)}-1=0=Q\\ \sqrt{(x-1)}-\sqrt{(x+1)}+1=0=R\\ \sqrt{(x-1)}-\sqrt{(x+1)}-1=0=S\end{array}\right\}$$

Of these, the only one that is satisfied by the value $x=\frac{5}{4}$ is R. The solution, therefore, of any one of these equations gives the same value of x, viz., $x=\frac{5}{4}$, and this value of x satisfies R only.

This whole subject of *derivation of equations* has been neglected by most writers on algebra in America until quite recently. The recent texts of Fisher and Schwatt, Beman and Smith, and several others have given considerable attention to the subject.

Professor Chrystal has given a very good treatment of the subject in his Algebra, Vol. I, § XIV. On page 285, he says, "There are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in a more slovenly fashion in elementary teaching."

This problem was also solved in a very excellent manner by *H. C. WHITAKER, G. B. M. ZERR, COOPER D. SCHMITT. W. H. CARTER, W. W. LANDIS, CHARLES C. CROSS, J. M. BOORMAN, J. D. CRAIG, A. F. KOVARIK, ELMER SCHUYLER*, and *J. SCHEFFER*.

110. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Put down any number of pounds, shillings and pence under £11, taking care that the number of pence is less than the number of pounds. Reverse this sum, putting pounds in the place of pence, and subtract from the original. Again reverse this remainder and add. The result in all cases will be £12 18s 11d, neither more nor less, whatever the amount with which we start. Verify and explain.

I. Solution by Prof. N. F. DAVIS, Brown University, Providence, R. I.; J. D. CRAIG, New Germantown, N. J.; the late SYLVESTER ROBBINS, North Branch Depot, N. J.; J. SCHEFFER, A. M., Hagerstown, Md.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; and H. C. WHITAKER, Ph. D., Manual Training School, Philadelphia, Pa.

	£	s.	d.
1st	a	b	c
Or thus	$a-1$	$19+b$	$12+c$
Reverse	c	b	a
Subtract	$a-c-1$	19	$12+c-a$
Reverse	$12+c-a$	19	$a-c-1$
Add	12	18	11

II. Solution by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; and G. B. M. ZERR, A.M., Ph.D., The Temple College, Philadelphia, Pa.

The restrictions placed on the problem are unnecessary. The proposition is true for any sum of pounds, shillings and pence, provided, when we subtract we take care to take the positive difference of the two sums obtained as described. The proposition may be proved thus:

Let a, b, c be a number of pounds, shillings and pence, respectively, and suppose $a>c$.

	£	s.	d.	
The two sums are	a	b	c	....(1).
	c	b	a	....(2).

Since $a>c$, before subtracting we must write (1) in the form

$$a-1 \qquad 19+b \qquad 12+c$$

Subtract (2), and we have

	$a-c-1$	19	$12+c-a$
Reverse	$12+c-a$	19	$a-c-1$
Add	£12	18s.	11d., the required result.

III. Solution by BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance, O., and W. W. LANDIS, A. M., Dickinson College, Carlisle, Pa.

Let x stand for the number of pounds, y the number of shillings, and z the number of pence. Also, let $x>z$ and $<z+12$, Of course, $y<20$, and $z<12$.

Then performing the operations as indicated, we have,

1.	£x	ys.	zd.
	£z	ys.	xd.
	£$(x-1-z)$	19s.	$(z+12-x)$d.
2.	£$(x-1-z)$	19s.	$(z+12-x)$d.
	£$(z+12-x)$	19s.	$(x-1-z)$d.
	£12	18s.	11d.

NOTE.—In the case of the decimal system of notation, under similar conditions we find the result to be always 1089. In general, in any system of notation in which n units of the first order make one of the second order, and m units of the second order make one of the third order; if, also, x stands for the number in the third order, y the second order, and z the first order, where $x>z$ and $<z+n$, then proceeding as indicated in the problem, we shall always find the result to be n units of the third order, $n-2$ units of the second order, and $n-1$ units of the first order.

IV. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics, Beaver College, Beaver, Pa.

This problem is an example of a class wherein the given quantities are caused to disappear by subtraction or cancellation—the resulting quantity depending on special circumstances, in this case the table ratios.

That the result will always be 12£ 18s 11d may be shown as follows:

	£	s.	d.
Given sum	a	b	c
Reversed	c	b	a
Remainder	$(a-1)-c$	19	$12+c-a$
Remainder reversed	$12+c-a$	19	$a-c-1$
Result	12	18	11

The sum need not be "under 11£" as was given in the problem, but up to a maximum of 23£ 19s 11d. More cannot be taken, as in the first subtraction we would have to "borrow" 2s instead of 1, or, in other words, use 24 as a ratio instead of 12.

If, however, we rigidly adhere to 20 and 12 as ratios, and use them, sums could be taken at random. As

£	s.	d.
100	30	25
25	30	100
74	19	−63
−63	19	74
12	18	11

A similar problem could be proposed with any table—the result varying with the constants of the table.

In general—if r and r' be the ratios (as 12 and 20) the result will be r of the highest, $r'-2$ of the next, and $r-1$ of the lowest denomination.

GEOMETRY.

135. Proposed by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

Solution by the PROPOSER.

Using quadrilinear notation, the equation to the circle circumscribing the quadrilateral whose sides are given by $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, is $\alpha\gamma=\beta\delta$ . . . . (1).

Now, it is well known that if the coördinates of one focus of a conic tangent to a given line be α', β', γ', δ', those of the other focus are proportional to $1/\alpha'$, $1/\beta'$, $1/\gamma'$, $1/\delta'$.

But by the problem, α', β', γ', δ' is on (1); then $\alpha'\gamma'=\beta'\delta'$ . . . . (2), or

$$\frac{1}{\alpha'\gamma'}=\frac{1}{\beta'\delta'} \quad \ldots\ldots (3).$$

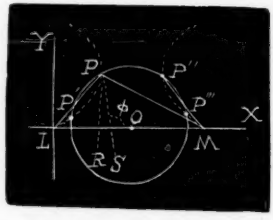
Substituting the reciprocals in (1) gives (3) also, and proves the theorem.

136. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Construct a triangle having given the base, the median line to the base, and the difference of the base angles.

I. Solution by B. L. REMICK, Instructor of Mathematics, Bradley Institute, Peoria, Ill.

Let $LM=a$=the base, $CP=m$=median to the base, $\alpha-\beta$=difference of base angles.



Then vertex P of required triangle lies on circle about C (mid point of LM) as center with radius m; it also lies on the locus of point of intersection of straight lines through L, M forming angles with base having the required constant difference. We propose to show that this latter locus is an equilateral hyperbola and that our problem has therefore four solutions corresponding to the four common points of the circle and hyperbola.

$\angle RPS$ between the perpendicular and angle bisector $=\frac{1}{2}(\alpha-\beta)$ by a well known result in geometry; and hence we have to consider the locus of intersection of two straight lines passing through two given points L, M so that the angle bisector remains parallel to itself.

Let coördinates of P be (x_1, y_1).

Equation PL is $y_1x-x_1y=0$.

Equation PM is $y_1x+(a-x_1)y-ay_1=0$.

Equation PS is $\dfrac{y_1x-x_1y}{\sqrt{(y_1^2+x_1^2)}}=\dfrac{y_1x+(a-x_1)y-ay_1}{\sqrt{[y_1^2+(a-x_1)^2}}$.

Since $\tan\varphi$ has a constant value we obtain

$$\frac{\dfrac{y_1}{\sqrt{[y_1^2+(a-x_1)^2]}}-\dfrac{y_1}{\sqrt{(y_1^2+x_1^2)}}}{\dfrac{x_1}{\sqrt{(y_1^2+x_1^2)}}+\dfrac{a-x_1}{\sqrt{[y_1^2+(a-x_1)^2]}}}=k \text{ (constant)}.$$

Simplifying, we have

$$(kx_1+y_1-ka)\sqrt{(y_1^2+x_1^2)}=(kx_1+y_1)\sqrt{[y_1^2+(a-x_1)^2]}.$$

Squaring and reducing, we obtain the form

$$2kx_1^2+2(1-k^2)x_1y_1-ky_1^2-2akx_1+(ak^2-a)y_1=0,$$

which shows the locus a conic.

Note points L, M, viz., $(0, 0)$, $(a, 0)$ are on locus.

Testing the discriminant to determine the nature of the conic, we have

$$AB-H^2=-4k^2-(1-k^2)^2=-(1+k^2)^2<0.$$

Hence the curve is a hyperbola.

To examine the locus farther by proceeding to standard form of equation, make use of transformtion $x=x'+x_0$, $y=y'+y_0$, equate to zero coefficients of x', y' in the equation found and obtain as first reduced form

$$4kx'^2+4(1-k^2)x'y'-4ky'^2-a^2k=0.$$

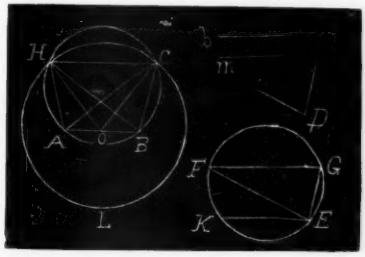
Again using formulæ $A'+B'=0$, $A'B'=-(1+k^2)^2$ obtain as final reduced equation

$$(1+k^2)x''^2-(1+k^2)y''^2=a^2k,$$

indicating an equilateral hyperbola.

Solved in a similar manner by *H. C. WHITAKER*.

II. Solution by G. B. M. ZERR, A.M., Ph.D., The Temple College, Philadelphia, Pa.; ALOIS K. KOVARIK, Decorah Institute, Decorah, Ia.; and J. SCHEFFER, A. M., Hagerstown, Md.



Let b=base, m=median, D=difference of the base angles.

In any circle $FGEG$, find a segment $FKEG$ containing an angle FEG equal to $\angle D$ and draw EK parallel to FG.

Take $AB=b$; with O the center of AB as center and a radius equal to m, describe the circle HCL. Construct the angles ABH and HBC equal to the angles KEF and FEG, respectively.

Through $ABCH$ describe a circle. Join AH, OH, OC, AC. Then ABC or ABH is the required triangle, since $AB=b$, $OH=OC=m$, $\angle HCB=\angle HAC=D$.

CALCULUS.

102. Proposed by G. B. M.ZERR. A.M., Ph.D., Professor of Chemistry and Physics.The Temple College.Philadelphia. Pa.

A right cone has its vertex at the focus of a paraboloid of revolution, the axis of the cone perpendicular to the axis of the paraboloid. Find the volume common to both.

Solution by the PROPOSER.

Let $x^2+y^2=c^2z^2$, be the equation to the cone. $y^2+z^2=4a(a+x)$, be the equation to the paraboloid.

The limits of z are $z=\frac{\sqrt{(x^2+y^2)}}{c}$ to $z=\frac{1}{2}\sqrt{(4a^2+4ax-y^2)}$;

of y, 0 and $\sqrt{\frac{4a^2c^2+4ac^2x-x^2}{1+c^2}}=y_1$;

of x, $2ac[c-\sqrt{(1+c^2)}]=x_2$ to $2ac[c+\sqrt{(1+c^2)}]=x_1$.

$$\therefore V=2\int_{x_2}^{x_1}\left(\frac{1}{2}\sqrt{(4a^2+4ax-y^2)}-\frac{\sqrt{(x^2+y^2)}}{c}\right)dxdy$$

$$=\int_{x_2}^{x_1}\left[\left[y_1\sqrt{[4a^2+4ax-y^2]}+4a(a+x)\sin^{-1}\left(\frac{y}{2\sqrt{[a(a+x)]}}\right)-\frac{y_1\sqrt{[x^2+y^2]}}{c}\right.\right.$$

$$\left.\left.-\frac{x^2}{c}\log[y+\sqrt{(x^2+y^2)}]\right]_0^{y_1}\right]dx=4a\int_{x_2}^{x_1}(a+x)\sin^{-1}\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}}{2\sqrt{[a(1+c^2)(a+x)]}}\right)dx$$

$$-\frac{1}{c}\int_{x_2}^{x_1}x^2\log\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}+c(2a+x)}{x\sqrt{(1+c^2)}}\right)dx.$$

$$\therefore V=\left[2a(a+x)^2\sin^{-1}\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}}{2\sqrt{[a(1+c^2)(a+x)]}}\right)\right.$$

$$\left.-\frac{x^3}{3c}\log\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}+c(2a+x)}{x\sqrt{(1+c^2)}}\right)\right]_{x_2}^{x_1}+a\int_{x_2}^{x_1}\frac{x(a+x)dx}{\sqrt{[4ac^2(a+x)-x^2]}}$$

$$+\tfrac{1}{3}\int_{x_2}^{x_1}\frac{x^3dx}{\sqrt{[4ac^2(a+x)-x^2]}}-\frac{1}{3c}\int_{x_2}^{x_1}x^2dx$$

$$-\frac{1+c^2}{3c}\int_{x_2}^{x_1}\frac{x^4\,dx}{\sqrt{[4ac^2(a+x)-x^2]}\{c(2a+x)+\sqrt{[4ac^2(a+x)-x^2]}\}}.$$

Let $2ac^2-x=2ac\sqrt{(1+c^2)}\cos\theta$.

$$\therefore\ V=2a^3c\int_0^\pi[c-\sqrt{(1+c^2)}\cos\theta][1+2c^2-2c\sqrt{(1+c^2)}\cos\theta]d\theta$$

$$+\tfrac{8}{3}a^3c^2\int_0^\pi[c-\sqrt{(1+c^2)}\cos\theta]^3d\theta-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^\pi[c-\sqrt{(1+c^2)}\cos\theta]^2\sin\theta d\theta$$

$$-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^\pi\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}.$$

$$\therefore\ V=\frac{2\pi a^3c^2}{3}(6+15c^2+10c^4)-\tfrac{16}{9}a^3c^2\sqrt{(1+c^2)}(1+4c^2)$$

$$-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^\pi\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}.$$

Let $c=\cot\beta$.

$$\therefore\int_0^\pi\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}=\frac{1}{\sin^3\beta}\int_0^\pi\frac{(\cos\beta-\cos\theta)^4d\theta}{1-\cos(\theta+\beta)}$$

$$=\frac{8}{\sin^3\beta}\int_0^\pi\sin^2\tfrac{1}{2}(\theta+\beta)\sin^4\tfrac{1}{2}(\theta-\beta)d\theta=D.$$

Let $\theta-\beta=2\varphi$. $\therefore\ \theta+\beta=2\varphi+2\beta$.

$$\therefore D=\frac{16}{\sin^3\beta}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}(\pi-\beta)}\sin^2(\varphi+\beta)\sin^4\varphi d\varphi$$

$$=\frac{16}{\sin^3\beta}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}(\pi-\beta)}(\sin^6\varphi\cos^2\beta+2\sin\beta\cos\beta\sin^5\varphi\cos\varphi+\sin^4\varphi\cos^2\varphi\sin^2\beta)d\varphi$$

$$=\tfrac{1}{2}\pi\operatorname{cosec}\beta(1+5\cot^2\beta)-\tfrac{2}{3}(1+4\cot^2\beta)=\tfrac{1}{2}\pi\sqrt{(1+c^2)}(1+5c^2)-\tfrac{2}{3}(1+4c^2).$$

$$\therefore\ V=\tfrac{8}{3}\pi a^3c^2(4+3c^2).$$

103. Proposed by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A park, in the shape of an ellipse whose diameters are 100 and 50 rods, respectively, is surrounded by a wall: one end of a rope, whose length is the circumference of the ellipse, is fastened (outside of the wall) at one end of the longer diameter and the other end at the other end of the same diameter. Over how much surface will a horse graze, which is fastened to a ring moving freely on the rope?

Remarks by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

No correct solution of this problem has yet been received. The problem has been misinterpreted by several of our contributors, who have mistaken it as

a corollary of Graves' Theorem. This was the interpretation according to which Dr. Haskell made his solution and which was inadvertently published in the MONTHLY, Vol. V, No. 4, page 111.

The problem as proposed is quite different and is not easy to solve, perhaps impossible.

The area grazed over consists of three parts :

(1) The part $ABP''P$—the area of this part is not difficult to find as the arc $PP'P''$ is the arc of a confocal ellipse; (2) two times the part $PEDAP$,—this part is generated by point, P, constrained to move under the composition of circular and elliptic evolutary motion ; and (3) two times the part $ADIA$, which is between the evolute of the elliptic field, the field, and the radius vector AD.

The part $PEDAP$ is the difficult part of the area to compute.

If any of our contributors will furnish a correct and complete solution of this problem it will be published in the next issue of the MONTHLY.

MECHANICS.

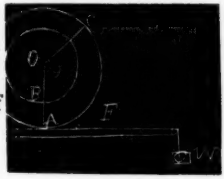
98. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, determine motion of spool and weight. [From problems in Mechanics at Harvard University.]

I. Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let T=tension on thread, x=distance spool's center moves, y=distance weight W moves, θ=angle AOC through which spool turns, $OA=a$, OB the radius of the thread$=b$.

Then $x=a\theta$, $y=(a-b)\theta$.



The forces acting on the spool are friction, F; tension, T; and reaction, R, perpendicular to the table.

Let m=mass of spool and thread, m_1=mass of weight. Then

$$m\frac{d^2x}{dt^2}=T-F,\ mk^2\frac{d^2\theta}{dt^2}=Fa\ldots(1,2),\text{ for spool};\ m_1\frac{d^2y}{dt^2}=T-m_1g\ldots(3),\text{ for weight.}$$

Eliminating F between (1, 2) we get

$$ma\frac{d^2x}{dt^2}+mk^2\frac{d^2\theta}{dt^2}=T\ldots.(4)\ ;\ \text{but}\ \frac{d^2x}{dt^2}=a\frac{d^2\theta}{dt^2},\ \frac{d^2y}{dt^2}=(a-b)\frac{d^2\theta}{dt^2}.$$

$\therefore\ m(a^2+k^2)\frac{d^2\theta}{dt^2}=T\ldots.(5)$, for spool; $m_1(a-b)\frac{d^2\theta}{dt^2}=T-m_1g\ldots(6)$, for weight.

Eliminating T between (5) and (6), we get

$$\frac{d^2\theta}{dt^2}=\frac{m_1g}{m(a^2+k^2)+m_1(b-a)}. \qquad \therefore\ T=\frac{mm_1(a^2+k^2)g}{m(a^2+k^2)+m_1(b-a)}.$$

$$\therefore\ m(a^2+k^2)\frac{d^2x}{dt^2}=\frac{amm_1(a^2+k^2)g}{m(a^2+k^2)+m_1(b-a)}$$

$$\text{or } \frac{d^2x}{dt^2}=\frac{am_1g}{m(a^2+k^2)+m_1(b-a)}$$

gives the acceleration of the spool. Also from (3),

$$m_1\frac{d^2y}{dt^2}+m_1g=\frac{mm_1(a^2+k^2)g}{m(a^2+k^2)+m_1(b-a)}. \qquad \therefore\ \frac{d^2y}{dt^2}=-\frac{m_1(b-a)g}{m(a^2+k^2)+m_1(b-a)},$$

gives acceleration of weight.

$\therefore$ If $b=a$, the weight remains at rest; if $a>b$, the weight ascends.

II. Solution by L. R. INGERSOLL, Student in Colorado College, Colorado Springs, Col., and the PROPOSER.

Let a be the radius of inner portion of spool, b radius of end. Then for rotation of spool about its center C we have the equation

$$\tfrac{1}{2}Mb^2\frac{d^2\theta}{dt^2}=Fb-mga\ldots.(1).$$

For motion of center of spool, the equation is

$$M\frac{d^2x}{dt^2}=mg-F\ldots.(2).$$

Eliminating F and remembering that $x=b\theta$, equations (1) and (2) become

$$\frac{d^2\theta}{dt^2}=\frac{2mg(b-a)}{3b^2M}\ldots.(3), \text{ and } \frac{d^2x}{dt^2}=\frac{2mg(b-a)}{3bM}\ldots.(4).$$

If $b>a$, the spool will roll towards weight with a constant angular acceleration, and its center will move with a constant acceleration b times its angular acceleration. But for same reason the string will be wound up with an acceleration a times its angular acceleration, and hence weight will descend with an acceleration $(b-a)$ times the angular acceleration.

If $b=a$, the spool will move towards or from the weight with constant velocity, and the weight will remain stationary.

If $b<a$, the spool will roll from the weight with constant angular acceler-

ation, and weight will descend with an acceleration $a-b$ times the angular acceleration.

99. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In a triangle ABC, base$=b$, area$=\triangle$, the principal moments of inertia at the centroid are $\frac{1}{72}m[a^2+b^2+c^2 \pm \sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}]$ and the principal axes at this point make with the base AC an angle θ given by

$$\tan 2\theta = \frac{4(c^2-a^2)\triangle}{(a^2-c^2)^2-b^2(a^2+c^2)+2b^4}.$$

Solution by the PROPOSER.

Let O be the centroid and transform from the rectangular axes Ox, Oy, to the oblique axes Ox, OB.

Also let $(b/6AD)(2AD-3y)=x'$.

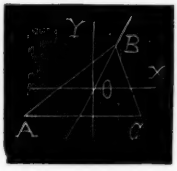
Then $\Sigma mxy=\rho\sin^2 D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x}^{x}(x+y\cos D)y\,dy\,dx=\frac{1}{36}b\rho AD^3\sin^2 D\cos D$

$$=\frac{1}{18}mAD^2\sin D\cos D.$$

$$\Sigma mx^2=\rho\sin D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x}^{x}(x+y\cos D)^2\,dy\,dx=\frac{1}{72}m(3b^2+4AD^2\cos^2 D).$$

$$\Sigma my^2=\rho\sin^3 D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x}^{x}y^2\,dy\,dx=\frac{1}{18}mAD^2\sin^2 D.$$

$$\therefore \tan 2\theta=\frac{4AD^2\sin 2D}{3b^2+4AD^2\cos 2D}.$$



But $\sin D=a\sin C/AD$, $\cos D=\frac{4AD^2+b^2-4a^2}{4b.AD}$.

$$\sin 2D=\frac{a\sin C(4AD^2+b^2-4a^2}{2b.AD^2}=\frac{2(c^2-a^2)\triangle}{b^2.AD^2}.$$

$$\cos 2D=\frac{AD^2-2a^2\sin^2 C}{AD^2}=\frac{2(a^2-c^2)-2b^2(a^2+c^2)+b^4}{4b^2.AD^2}.$$

$$\therefore \tan 2\theta=\frac{4(c^2-a^2)\triangle}{(a^2-c^2)^2-b^2(a^2+c^2)+2b^4}.$$

$A\cos^2\theta+B\sin^2\theta=\frac{1}{72}m(3b^2+4AD^2\cos^2 D).$

$A\sin^2\theta+B\cos^2\theta=\frac{1}{18}mAD^2\sin^2D.$

$\therefore\ A+B=\frac{1}{72}m(3b^2+4AD^2).$

$(A-B)\cos2\theta=\frac{1}{72}m(3b^2+4AD^2\cos2D).$

$\therefore\ A=\frac{1}{72}m[a^2+b^2+c^2+2\sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}].$

$B=\frac{1}{72}m[a^2+b^2+c^2-2\sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}].$

MISCELLANEOUS.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

II. Solution by M. A. GRUBER, A.M., War Department, Washington, D. C.; HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; and the PROPOSER.

Rational right triangles are divided into three kinds: (1), prime integral; (2), composite integral; and (3), fractional.

It is evident that prime integral right triangles are the basis of work.

To produce a prime integral right triangle we take *two integers prime to each other, one odd and the other even.* Then, twice their product will give one of the legs, the difference of their squares will give the other of the legs, and the sum of their squares will be the hypotenuse; or taking p and q as the two integers, the respective sides are $2pq$, p^2-q^2, and p^2+q^2.

This process will produce any prime integral right triangle.

The squares of even numbers end in 4, 6, and 0, and of odd numbers in 1, 9, and 5.

When the two prime integers, or p and q end, respectively, in 0 and an odd figure, or in 5 and an even figure, their product, or pq of the side $2pq$ contains the factor 5; the other sides being odd numbers *not* ending in 5.

When the squares of the two prime integers, or p^2 and q^2 end, respeetively, in 1 and 6 or in 9 and 4, the difference of these squares, or the side p^2 and q^2 ends in 5, and is, therefore, divisible by 5.

When the squares of the two prime integers end, respectively, in 1 and 4 or in 9 and 6, the sum of these squares, or the side p^2+q^2 ends in 5.

The above contains all the combinations of any two prime integers, one odd and the other even, according to the formation of the sides of prime integral right triangles.

$\therefore$ In every *prime* integral right triangle one of the sides and *only one* is divisible by 5.

In composite integral and fractional right triangles either *one* or *all* of the sides must be divisible by 5. For, in order to have more than one side divisible by 5, the highest common factor of the three sides must contain the factor 5.

$\therefore$ In rational right triangles either *one* or *all* of the sides will be divisible by 5.

86. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the payment on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

Solution by WM. FRED FLEMING, Principal of High School, Denison, Texas.

One general formula for finding the number of fixed monthly payments required to extinguish a debt which bears interest is as follows:

$$\text{Number of payments}=\frac{\log\left(\frac{p}{p-mP}\right)}{\log(m+1)}$$

where p=payment, P=principal, and m=fraction of principal accruing as monthly interest.

In the given problem the number of payments is equal since the notes are to be lifted at the same time. Hence

$$\frac{\log\left(\frac{p}{p-10.50}\right)}{\log(1\frac{7}{200}+1)}=\frac{\log\left(\frac{25-p}{25-p-9.50}\right)}{\log(2\frac{1}{100}+1)}.$$

Whence, using 7-figure logarithms,

$$\frac{p}{p-2\frac{1}{2}}=\left(\frac{25-p}{3\frac{1}{2}-p}\right)^{1.1662}$$

which equation, solved by double position (or some other method of approximation) gives, as the nearest payments involving *even cents*, \$12.74 for the payment on the \$1800 note, and \$12.26 as payment on the \$1900 note.

Number of payments=298.9, or 24 years and 11 months (nearly) will be required to lift the note.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

What is the size of the smallest cubical box, inside dimension, that will contain four balls each ten inches in diameter?

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

At the corners of a triangle sides a, b, c feet are towers d, e, f feet high. At what point must a ladder be placed so that it will just reach to the top of each tower without moving? How long is the ladder? Substitute $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$; d, e, f at A, B, C, respectively.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

ALGEBRA.

125. Proposed by LESLIE L. LOCKE, Instructor in Mathematics, Michigan Agricultural College, Ingram County, Mich.

What special expedient will solve the system

$$\left.\begin{aligned} x^4-y^4&=369 \\ x-y&=1 \end{aligned}\right\}\ ?$$

126. Proposed by CHARLES C. CROSS, Meredithville, Va.

A and B run a race; B, who runs slower than A by a miles in b hours, starts first by c minutes, and they get to the n-mile stone together; required their rates of running. If $a=1$, $b=2$, $c=2$, and $n=4$, what is the result?

*** Solutions of these problems should be sent to J. M. Colaw not later than Jan. 10.

GEOMETRY.

154. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

The angle between the edge of a trihedral angle and the bisector of the opposite face angle is less than, equal to, or greater than, half the sum of the other two face angles, according as it is itself acute, right, or obtuse.

155. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Tex.

A special case of the following problem was sent me some time ago by an ex-member of one of my engineering classes, as occuring on the Southern Pacific Ry. near Devil's River:

Two straight tracks, p feet between centers, are to be united by a cross-over composed of two curves of radius R, and a length L of intervening tangent. Required the central angles and the distance between tangent points, measured along main track. In the special case referred to p was 62 feet, L 100 feet with 9° 30′ curves.

156. Proposed by F. M. McGAW, A. M., Professor of Mathemstics, Bordentown Military Institute, Bordentown, N. J.

To construct an equilateral triangle such that its vertices shall be in each of two parallel lines and a point fixed between these lines.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

CALCULUS.

116. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

"Prove that the length of the *greatest* beam of square section that can be cut from a log l feet long and in the shape of a conic frustum, diameters D and d, is $\frac{1}{3}lD \div (D-d)$ feet."

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A frustum of a paraboloid of revolution closed at both ends has a given volume. Find its interior dimensions when its surface is a minimum.

118. Proposed by J. W. YOUNG, Oliver Graduate Student, Cornell University, Ithaca, N. Y.

Find the differential equations of the system of parabolas, $y^2=4a^2(x+a^2)$, and of its orthogonal trajectories, and interpret the result. Find also the equation of the system of trajectories.

*** Solutions of these problems should be sen to J. M. Colaw not later than Jan. 10.

MECHANICS.

105. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A man, weighing 150 pounds at the surface of the earth, ascends in a balloon, until the area visible to him is $2\pi R^2(1-\frac{1}{11}\ 2)$. What is his weight at that height?

106. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Vary the radius of curvature of a plane curve inversely as the abscissa; then the solution will give you, (1) Ryan's Equation of the Elastic Curve, and (2) Wood's Equation of the Hydrostatic Curve.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

AVERAGE AND PROBABILITY.

97. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A straight line is drawn at random across a circle, and five points are taken at random in the surface of the circle. Required the chance that all the points are on the same side of the line.

98. Proposed by REV. PREBENDARY WHITWORTH, A. M.

A has £m and B has £n. They play for points until one of them has lost all his money. If α and β be the respective chances that A and B win any point, the expectation of the number of points played will be

$$\frac{n\alpha^n(\alpha^m-\beta^m)-m\beta^m(\alpha^n+\beta^n)}{(\alpha-\beta)(\alpha^{m+n}-\beta^{m+n})}.$$

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

MISCELLANEOUS.

98. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A particle describes an ellipse under an attraction always directed to the vertex; to determine the law of the attraction.

99. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Through the zenith of an observer at the sea-coast in north latitude $\varphi=40°$, a "cat's-tail cloud," height $h=10$ miles, extends northeast until it touches the horizon. How far from the observer is the *advance-end* of the cloud? What is the length of the cloud measured from the end specified to the observer's zenith?

*** Solutions of the problems should be sent to J. M. Colaw not later than Jan. 10.

EDITORIALS.

Prof. B. O. Peirce, of Harvard University, has been granted a year's leave of absence.

Dr. G. A. Bliss has been appointed Instructor in Mathematics at the University of Minnesota.

Prof. Irvin Stringham has returned to the University of California after a year of travel in Europe.

Prof. W. Dyck has been appointed Director of the Technical High School in Munich for the next three years.

Prof. R. E. Allardice, of Leland Stanford Jr. University, is spending a year abroad on leave of absence.

Prof. E. J. Townsend has returned to the University of Illinois, after two years of study at the University of Göttingen.

Owing to our engravers having sent the plate of Mr. Tucker's portrait to some other Springfield, and which has not yet been returned to us, we are obliged to have it appear in next issue.

BOOKS AND PERIODICALS.

Logarithmic and Trigonometric Tables, Five-Place and Four-Place. Edited by D. A. Murry, Cornell University. 8vo. Cloth, 96 pages. Price, 54 cents. New York: Longmans, Green & Co.

This book contains Five-Place Logarithms of Numbers; Five-Place Logarithms of the sine, cosine, tangent and cotangent for each minute from 0° to 90°; also Four-Place Logarithms corresponding to the Five-Place Logarithms of Number and the Trigonometric Functions. The editor believes that the proportional parts should be calculated, and not copied, by those who use logarithms and trigonometric tables for the first time. So these tables do not have the proportional parts set down in the tables. We heartily concur with him in this opinion. B. F. F.

Notes on the History of American Text-Books on Arithmetic. By J. M. Greenweed, LL. D., Superintendent of Schools, Kansas City, Mo., and Artemas Martin, A. M., Ph. D., LL. D., Washington, D. C. Issued by the National Bureau of Education.

The pamphlet before us is a chapter from the Report of the Commissioner of Education, and comprises pages 781—838 of the Report. The titles of over 250 text-books on Arithmetic are given, together with the names of the authors. The earliest published Arithmetic in the list here given is that of Reffelt's, published in 1861. Earlier Arithmetics are given in a previous chapter. In many instances short biographical sketches of the authors are given. This bibliography will, when complete, be of great value. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

The world has never known a more dramatic situation than that presented by the foreign community within the walls of Peking while cut off from communication with their countrymen. During these long, doubtful weeks, the most interesting figure in this international tragedy was Sir Robert Hart, who for more than twenty-five years has been as far, as a European might, the statesman guiding the affairs of the Chinese Empire. Those familiar in any degree with Eastern conditions hoped, after the relief of Peking, that Sir Robert would break his long rule of silence and give to the world his story of the events which led to the closing of the gates of the British Legation, and his views as to the policies which should prevail in the settlement of difficult questions which have arisen. On the 17th of October, the following cable-message from Sir Robert's London representative to the editor of *The Cosmopolitan* was received: "Sir Robert Hart has sent for November number *Fortnightly* London, and *Cosmopolitan*, New York, an important article on siege of Peking, about fifteen thousand words, which I will post you to-morrow."

The MS. arrived in time to be included in the December issue. It will be read with the deepest interest, both by statesmen and the general public. *The Cosmopolitan* has been highly honored by Sir Robert Hart in his selection of the American magazine through which this valuable contribution to the history of the world is given publicity. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine, edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., New York.

The November number contains interesting articles on the election. In addition, the following valuable articles are to be found in it: The Great Growth of Trusts in England, The Political Beginning in Porto Rico, and the Hall of Fame for Great Americans.

B. F. F.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

This magazine gives a brief resume of all the principal events of the month, which take place anywhere in the civilized world. It is just the journal for the busy man.

B. F. F.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Mo., as second-class matter.

VOL. VII. DECEMBER, 1900. No. 12.

SOME METHODS OF CONSTRUCTING SUBSTITUTION GROUPS.

By DR. G. A. MILLER, Cornell University, Ithaca, N. Y.

We have observed that the first n powers of a circular substitution of degree n constitute a group of order n. If a substitution is composed of more than one cycle its order is the least common multiple of the degrees of all its cycles. This is also the order of the group which it generates. Any group which is generated by a single substitution is called *cyclical*. A cyclical group is clearly abelian but an abelian group is not necessarily cyclical. *There is one and only one cyclical abstract group of every possible order n* since the combinatory laws of such a group are the same as those of the roots of the equation $x^n=1$, where n is any positive integer. The only orders of which there is no group except the cyclical are those which are not divisible by the square of a prime and do not contain any prime factor which is congruent to 1 with respect to another prime factor as modulus. The composite orders below 100 that have this property are 15, 33, 35, 51, 65, 69, 77, 85, 87, 91, 95.

Let S and T represent any two commutative substitutions. This property is expressed by each of the following three equivalent equations :

$$ST=TS \qquad T^{-1}ST=S \qquad T=S^{-1}TS$$

where T^{-1} and S^{-1} indicate the inverse of T and S, respectively; *i. e.* $T^{-1}T=1=S^{-1}S$. In general $T^{-1}ST$ is called the transform of S with respect to T. All the substitutions of degree m that are commutative with any given substitution T

of degree n form a group, for if S_1 and S_2 are commutative with T then will S_1^2, S_2^2, S_1S_2, S_2S_1 have the same property.* By raising both members of the equation $S^{-1}TS=T$ to the β power we have

$$S^{-1}TS.S^{-1}TS.S^{-1}TS\ldots.=T^\beta \text{ or } S^{-1}T^\beta S=T^\beta .$$

Hence every substitution that transforms T into itself must transform each of the substitutions of the group generated by T into itself. When T consists of a single cycle m cannot be less than n,† and when $m=n$ all the substitutions that are commutative with T are the n different powers of T; for if some other substitution of degree n would transform T into itself its product into some power of T would give a substitution of degree less than n that would also transform T into itself.

If T consists of β cycles of degree $n\div\beta$ (in this case T is said to be a regular substitution) the order of the group of degree n which is formed by all the substitutions that are commutative with T is $\left(\frac{n}{\beta}\right)^\beta.\beta!$ since the β cycles are permuted according to the symmetric group of degree β. In general, let T contain a_1 cycles each of degree a_2, b_1 cycles each of degree b_2, ...; then the order of the group which is formed by all the substitutions of degree n that are commutative with T is $a_2^{a_1}.a_1!.b_2^{b_1}.b_1!\ldots.$ In particular, when no two cycles of T are of the same degree the order of this group is the product of the degrees of these cycles, and the necessary and sufficient condition that the powers of T include all the substitutions of degree n that are commutative with T is that the degree of each cycle of T is prime to that of every other cycle. When $m=n+\alpha$ ($\alpha>1$) the required group is obtained by multiplying the given group of degree n by the symmetric group of order $\alpha!$

We have now considered the groups which are formed by all the substitutions which are commutative with a given substitution T; *i. e.* by all the values of S which satisfy the equation $S^{-1}TS=T$. This is a special case of the problem to find all the groups which are formed by all the values of S which satisfy the equation $S^{-1}TS=T$.

By raising both members of this equation to the β power we obtain $S^{-1}T^\beta S=T^\beta=(T^\beta)^\alpha$; *i. e. if a substitution transforms T into a certain power it transforms all the substitutions of the group generated by T into the same power*, and hence it must transform this group into itself. Since all the substitutions which transform a group into itself must form a group all the values of S which satisfy the equation $S^{-1}TS=T^\alpha$, α having every possible value, constitute a group (G) whose order we proceed to determine.

From $(S^{-1}TS)^\alpha=S^{-1}T^\alpha S$ and $S^{-1}\,|\,S=1$ it follows that a transform of T cannot be of a higher order than T, and from $S^{-1}T^\alpha S=1$ we have $T^\alpha S=S$ or

*Cf. Burnside, *Theory of Groups*, 1897, page 215.

†It will be assumed throughout that the substitutions of degree m which are commutative with T involve no elements except those contained in T whenever m is equal to or less than n. When m is greater than n these substitutions are supposed to include all the elements of T.